

# Least Capacity Point of Triangles

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**ABSTRACT.** Let  $D$  be a compact convex domain in the plane. Pólya & Szegő and, independently, Levi & Pan defined the point  $p \in D$  that is “best insulated from the boundary  $C$  of  $D$ ”. We compute  $p$  in the case when  $C$  is an isosceles right triangle, revisiting exact results from the study of complex conformal mappings.

Let  $T = \{x + iy \in \mathbb{C} : x > 0, y > 0, x + y < 1\}$ , the interior of a right isosceles triangle. Let  $\Delta$  denote the unit disk in  $\mathbb{C}$ . Let  $w \in T$ . The Riemann mapping theorem guarantees the existence of a conformal map  $f_w : T \rightarrow \Delta$  such that  $f_w(w) = 0$ . Moreover  $f_w$  extends continuously to the boundary of  $T$ , mapping it homeomorphically onto the unit circle.

Pólya & Szegő [1] wrote about the **inner radius**  $r_w$  of  $T$  **relative to**  $w$ ; the point  $w$  that maximizes  $r_w$  is the same as the **least capacity point** characterized sixty years later by Levi & Pan [2]. It is sufficient to examine the ratio  $f_w(z)/(z - w)$  or, more precisely, to minimize the difference

$$\underbrace{\ln |f_w(z)|}_{\text{Green's function for } T} - \ln |z - w|$$

in the limit as  $z \rightarrow w$ , over all  $w \in T$ . The first term is harmonic in  $T - \{w\}$ , is bounded outside every neighborhood of  $w$ , and vanishes at the boundary [3]; the additional term serves to repair the singularity at  $w$ .

Our purpose is to find an explicit expression for  $f_w$ . We begin with a beautiful formula communicated in [4]. This is followed by commentary regarding specific examples in the literature. We conclude with a second approach, drawing upon results in [5, 6].

The quantity  $r_w$  is the same as the **conformal radius** relative to  $w$  in our simple setting (involving triangles only).

Two addenda have emerged over time, one concerning the 30°-60°-90° triangle and the other concerning the (fairly arbitrary) 6-9-13 triangle. We wonder whether a least capacity point qualifies as a *triangle center* [7]. The present paper, in part, continues our discussion [8] in which three other candidate triangle centers were examined.

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## 1. WEIERSTRASS SIGMA

Using the Schwarz reflection principle,  $f_w$  extends across the hypotenuse of  $T$  to an analytic function (still called  $f_w$ ) on the square  $\{x + iy \in \mathbb{C} : 0 < x < 1, 0 < y < 1\}$  except for a simple pole at  $w' = 1 + i - i\bar{w}$ . Using the Schwarz reflection principle twice more,  $f_w$  extends to the doubled square  $\{x + iy \in \mathbb{C} : -1 < x < 1, -1 < y < 1\}$  with zeroes at  $\{\pm w, \pm \bar{w}'\}$  and poles at  $\{\pm w', \pm \bar{w}\}$ . Clearly  $f_w(-1 + iy) = f_w(1 + iy)$  and  $f_w(x - i) = f_w(x + i)$  always. Let  $\Lambda$  denote the lattice  $\{2m + 2ni : m, n \in \mathbb{Z}\}$ . We may extend  $f_w$  to a doubly periodic meromorphic function on  $\mathbb{C}$  via  $f_w(z + \lambda) = f_w(z)$  for all  $\lambda \in \Lambda$ . Assume that  $f_w(1) = 1$  without loss of generality. We deduce that [9]

$$f_w(z) = C_w \frac{\sigma(z - w)\sigma(z + w)\sigma(z - \bar{w}')\sigma(z + \bar{w}')}{\sigma(z - w')\sigma(z + w')\sigma(z - \bar{w})\sigma(z + \bar{w})} \quad \text{for all } z \in T$$

where

$$\sigma(z) = z \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right),$$

$$C_w = \frac{\sigma(1 - w')\sigma(1 + w')\sigma(1 - \bar{w})\sigma(1 + \bar{w})}{\sigma(1 - w)\sigma(1 + w)\sigma(1 - \bar{w}')\sigma(1 + \bar{w}')}.$$

In order to employ Mathematica (or other computer algebra package), the half-periods 1,  $i$  give rise to invariants

$$g_2 = \frac{1}{256\pi^2} \Gamma\left(\frac{1}{4}\right)^8 = 11.8170450080\dots, \quad g_3 = 0$$

which must be passed to the software implementation of  $\sigma$ . As  $z \rightarrow w$ , the ratio  $|f_w(z)/(z - w)|$  simplifies to

$$h(w) = \left| \frac{\sigma(1 - w')\sigma(1 + w')\sigma(1 - \bar{w})\sigma(1 + \bar{w})}{\sigma(1 - w)\sigma(1 + w)\sigma(1 - \bar{w}')\sigma(1 + \bar{w}')} \cdot \frac{\sigma(2w)\sigma(w - \bar{w}')\sigma(w + \bar{w}')}{\sigma(w - w')\sigma(w + w')\sigma(w - \bar{w})\sigma(w + \bar{w})} \right|$$

because  $\sigma(z - w)/(z - w) \rightarrow 1$ . Further simplification does not seem possible. Numerical minimization gives the least capacity point to be  $w_0 = (1 + i)t_0$ , where

$$t_0 = 0.3011216108413220815538254\dots$$

No closed-form expression for  $t_0$  is apparent, at least not here. We observe (to high precision) that

$$\frac{1}{h(w_0)} = 0.3346161009568417919464744\dots = \frac{4\sqrt{2\pi}}{3^{3/4}} \Gamma\left(\frac{1}{4}\right)^{-2}$$

which is encouraging since the latter is the maximum inner radius for  $T$  [1]. A rigorous proof, however, remains open.

## 2. WEIERSTRASS P

The zero  $w_0$  in the preceding section identified a conformal map  $f_{w_0} : T \rightarrow \Delta$  that is, in particular, extremal in some sense. Any map onto the upper half plane  $\mathbb{C}^+$  can be easily recast as a map onto the disk  $\Delta$  (via composition with a linear fractional transformation). As a slight detour, let us similarly identify other better-known conformal maps  $T \rightarrow \mathbb{C}^+$  that have appeared in the literature.

Two maps  $\varphi, \psi$  are prescribed to take the following values on the vertices of  $T$ :

$$(0, 1, i) \xrightarrow{\varphi} (0, 1, \infty),$$

$$(0, 1, i) \xrightarrow{\psi} (\infty, 0, 1).$$

The Schwarz-Christoffel transformation gives [10, 11, 12]

$$\varphi^{(-1)}(\zeta) = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 B\left(\zeta, \frac{1}{2}, \frac{1}{4}\right),$$

$$\psi^{(-1)}(\zeta) = \frac{i-1}{\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 B\left(\zeta, \frac{1}{4}, \frac{1}{4}\right) + 1$$

where

$$B(\zeta, \alpha, \beta) = \int_0^\zeta s^{\alpha-1} (1-s)^{\beta-1} ds$$

is the incomplete Euler beta function. Although expressions for  $\varphi^{(-1)}, \psi^{(-1)}$  are famous, their inverses are comparatively obscure. Geyer [13] provided

$$\psi(z) = -\frac{1}{4\wp(1)} \frac{(\wp(z) - \wp(1))^2}{\wp(z)}$$

where

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left( \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

$$\wp(1) = \frac{1}{32\pi} \Gamma\left(\frac{1}{4}\right)^4 = 1.7187964545\dots$$

Choose the linear fractional transformation  $\mathbb{C}^+ \rightarrow \Delta$  to be  $\zeta \mapsto (\zeta - i)/(\zeta + i)$ . Hence we wish to solve the equation  $\psi(w) = i$ , but this is immediately seen to yield

$$w = \frac{i-1}{\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 B\left(i, \frac{1}{4}, \frac{1}{4}\right) + 1 = 0.2970894700\dots + (0.1926647354\dots)i.$$

It is clear that

$$\overline{\psi(i\bar{z})} \cdot \varphi(z) = 1$$

and we wish to solve the equation  $\varphi(w) = i$ , but this too is immediately seen to yield

$$w = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 B\left(i, \frac{1}{2}, \frac{1}{4}\right) = 0.1926647354... + (0.2970894700...)i.$$

A representation of arbitrary  $f_w$  in terms of  $\wp$  (analogous to our representation in terms of  $\sigma$ ) is also possible [9], but we haven't pursued this.

### 3. JACOBI ELLIPTIC

Let

$$F[\phi, m] = \int_0^{\sin(\phi)} \frac{d\tau}{\sqrt{1-\tau^2} \sqrt{1-m\tau^2}}$$

denote the incomplete elliptic integral of the first kind and  $K[m] = F[\pi/2, m]$ ; we purposefully choose formulas here to be consistent with Mathematica. The three basic Jacobi elliptic functions are defined via

$$\begin{aligned} u &= \int_0^{\text{sn}(u,m)} \frac{d\tau}{\sqrt{1-\tau^2} \sqrt{1-m\tau^2}} = \int_{\text{cn}(u,m)}^1 \frac{d\tau}{\sqrt{1-\tau^2} \sqrt{m\tau^2 + (1-m)}} \\ &= \int_{\text{dn}(u,m)}^1 \frac{d\tau}{\sqrt{1-\tau^2} \sqrt{\tau^2 - (1-m)}} \end{aligned}$$

and we shall require all three of these. Define

$$\kappa = \frac{K[1/2]}{\sqrt{2}} = \frac{1}{2^{5/2}\sqrt{\pi}} \Gamma\left(\frac{1}{4}\right)^2 = 1.3110287771... = \frac{1.8540746773...}{\sqrt{2}}$$

and let  $\tilde{T} = \{x + iy \in \mathbb{C} : y > 0, y < x + \kappa, y < -x + \kappa\}$ . As in Section 2, let us first examine a conformal map  $\theta : \tilde{T} \rightarrow \mathbb{C}^+$  which takes prescribed values on the vertices of  $\tilde{T}$ :

$$(-\kappa, \kappa, i\kappa) \xrightarrow{\theta} (-1, 1, \infty).$$

The Schwarz-Christoffel transformation gives [14]

$$\theta^{(-1)}(\zeta) = \frac{1}{2} \int_0^\zeta \frac{ds}{(1-s^2)^{3/4}} = \frac{\zeta}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, \zeta^2\right)$$

which involves the following Gauss hypergeometric function:

$${}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, z\right) = \frac{\Gamma(1/4)}{2\sqrt{2}\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n+3/4)}{\Gamma(n+3/2)} \frac{z^n}{n!}.$$

Although the expression for  $\theta^{(-1)}$  is famous, its inverse is comparatively obscure. Kober [5, 6] provided

$$\theta(z) = \sqrt{2} \operatorname{sn}\left(\sqrt{2}z, \frac{1}{2}\right) \operatorname{dn}\left(\sqrt{2}z, \frac{1}{2}\right).$$

Choose the linear fractional transformation  $\mathbb{C}^+ \rightarrow \Delta$  to be  $\zeta \mapsto (\zeta - i)/(\zeta + i)$ . Thus we wish to solve the equation  $\theta(w) = i$ , but this is immediately seen to yield

$$w = \frac{i}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{2}, -1\right) = (0.4154481080\dots)i = (0.3168871006\dots)\kappa i.$$

Now let  $w \in \tilde{T}$  be arbitrary. Define  $f_w : \tilde{T} \rightarrow \Delta$  for which  $f_w(w) = 0$  via [12]

$$\begin{aligned} f_w(z) &= -\frac{\frac{\theta(z)-i}{\theta(z)+i} - \frac{\theta(w)-i}{\theta(w)+i}}{1 - \left(\frac{\theta(z)-i}{\theta(z)+i}\right) \overline{\left(\frac{\theta(w)-i}{\theta(w)+i}\right)}} \\ &= \frac{\theta(z) - \theta(w)}{\theta(z) - \overline{\theta(w)}} \end{aligned}$$

(following the construction of Green's function in [15], but beware of misprints). As  $z \rightarrow w$ , the ratio  $|f_w(z)/(z - w)|$  simplifies to

$$h(w) = \left| \frac{\theta'(w)}{\theta(w) - \overline{\theta(w)}} \right|$$

where  $\theta'$  denotes the derivative of  $\theta$ . Numerical minimization gives the least capacity point to be

$$\begin{aligned} \tilde{w}_0 &= (0.3977567783173558368923490\dots)\kappa i \\ &= (1 - 2t_0)\kappa i \end{aligned}$$

as expected, since

$$\frac{\kappa - |\tilde{w}_0|}{\sqrt{2}\kappa} = \frac{\sqrt{2}t_0}{1}$$

by the similarity of triangles  $\tilde{T}$  and  $T$ . Restricting attention to the  $y$ -axis only, we have

$$h(iy) = \frac{i \operatorname{cn} \left( i\sqrt{2}y, \frac{1}{2} \right)^3}{\sqrt{2} \operatorname{sn} \left( i\sqrt{2}y, \frac{1}{2} \right) \operatorname{dn} \left( i\sqrt{2}y, \frac{1}{2} \right)}.$$

Differentiating with respect to  $y$  and setting the result equal to zero, the equation

$$\operatorname{dn} \left( i\sqrt{2}y, \frac{1}{2} \right) = \sqrt{\frac{1 + \sqrt{3}}{2}}$$

is found, therefore

$$t_0 = \operatorname{Re} \left\{ \frac{1}{2\kappa} F \left[ \arcsin \sqrt{\frac{1 + \sqrt{3}}{2}}, 2 \right] \right\} = 0.3011216108413220815538254...$$

is the sought-after closed-form expression. Such an outcome was not apparent in Section 1.

An old paper by Love [16] discusses conformal maps on four exceptional triangles (including the isosceles right triangle) and utilizes the  $\wp$  function; unfortunately we haven't succeeded in following the details. Two other papers [17, 18], despite promising titles, evidently assess Green's function created for different settings (non-Laplacian) than ours.

#### 4. ADDENDUM: 30°-60°-90° TRIANGLE

Without entering any lengthy explanations, let

$$\kappa = \frac{1}{2^{5/3}\sqrt{\pi}} \Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{1}{6} \right) = \frac{5.2999162508...}{2}$$

and define  $T = \{x + iy \in \mathbb{C} : x > 0, y > 0, \sqrt{3}x + y < \sqrt{3}\kappa\}$ . The conformal map  $\theta : T \rightarrow \mathbb{C}^+$  taking prescribed values on the vertices of  $T$ :

$$(0, \kappa, i\sqrt{3}\kappa) \xrightarrow{\theta} (0, 1/4, \infty)$$

is given by

$$\theta(z) = \frac{3\sqrt{3} \operatorname{sn} \left( \frac{2^{2/3}}{3^{3/4}}z, \frac{2 + \sqrt{3}}{4} \right)^2 \operatorname{dn} \left( \frac{2^{2/3}}{3^{3/4}}z, \frac{2 + \sqrt{3}}{4} \right)^2}{\left\{ 1 + \operatorname{cn} \left( \frac{2^{2/3}}{3^{3/4}}z, \frac{2 + \sqrt{3}}{4} \right) \right\}^4}.$$

(We could not decipher pp. 186–187 of [5], hence we turned to pp. 184–185 and made necessary adjustments.) For example,  $\theta(w) = i$  occurs when

$$\begin{aligned} w &= 3 \left( \frac{1}{2} + \frac{1+i}{\sqrt{2}} \right)^{1/3} {}_2F_1 \left( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{2} + \frac{1+i}{\sqrt{2}} \right) - \kappa \\ &= 0.7065812599\dots + (1.6814450943\dots)i \\ &= (0.2666386510\dots)\kappa + (0.6345176092\dots)\kappa i. \end{aligned}$$

Now let  $w \in T$  be arbitrary. Define  $f_w : T \rightarrow \Delta$  for which  $f_w(w) = 0$  as before; the ratio  $|f_w(z)/(z-w)|$  tends to

$$h(w) = \left| \frac{\theta'(w)}{\theta(w) - \overline{\theta(w)}} \right|$$

as  $z \rightarrow w$ . Numerical minimization gives the least capacity point to be

$$\begin{aligned} w_0 &= (0.3599371272406945147550792\dots)\kappa + \\ &\quad (0.4062604057445303763104149\dots)\kappa i. \end{aligned}$$

We have not attempted to find a closed-form expression for  $w_0$ . To high precision,

$$\frac{1}{h(w_0)} = (0.2105704622445114724079460\dots)(2\kappa) = \frac{2^{4/3}\pi}{5^{5/12}}\Gamma\left(\frac{1}{3}\right)^{-3}(2\kappa)$$

which is the (corrected) maximum inner radius for  $T$  [1].

## 5. ADDENDUM: 6-9-13 TRIANGLE

No such exact formulas can be found for arbitrary triangles. The Schwarz-Christoffel toolbox for Matlab [19, 20], coupled with the Optimization toolbox, makes numerical computations of least capacity points readily accessible. Recall that we wish to assess whether such points can be treated as triangle centers. For the triangle with vertices

$$0, \quad 6, \quad -\frac{13}{3} + \frac{4\sqrt{35}}{3}i$$

the following code:

```
function q = arbitra(w)
p = polygon([0 6 -13/3+(4*sqrt(35)/3)*i])
f = diskmap(p,scmapopt('Tolerance',1e-18))
f = center(f,w(1)+i*w(2));
p = parameters(f);
q = -abs(p.constant);
```

gives (for example) that the inner radius at the centroid is

$$-\text{arbitra} \left( \frac{5}{9} + \frac{4\sqrt{35}}{9}i \right) = 1.802305....$$

Using the centroid as a starting guess, we solve a constrained minimization problem as follows:

```
format long
options=optimset('Algorithm','interior-point','TolCon', 1e-15);
A = [-4*sqrt(35) -13; 0 -1; 4*sqrt(35) 31]
b = [0 0 24*sqrt(35)]
v0 = [5/9 4*sqrt(35)/9];
[v,fv] = fmincon(@arbitra,v0,A,b,[],[],[],[],[],options);
```

yielding the maximum inner radius to be 1.979479... and the corresponding least capacity point to be  $0.929617... + (1.842564...)i$ . This particular triangle serves as a benchmark in [7] to distinguish various centers. The imaginary part is the perpendicular distance from the proposed center to the shortest triangle side. Since the numerical value 1.842... does not appear in the database, we infer that this center is new.

Figures 1, 2, 3 provide conformal map images of ten evenly-spaced concentric circles in the disk. These are optimal in the sense that their center is “best insulated” from the triangle boundary. Orthogonal trajectories are also indicated.

The literature on this subject is larger than we originally thought. The phrase **conformal center** is sometimes used to denote what we call the least capacity point. (This is not to be confused with a different sense of the same phrase in [19, 20].) Some discussion of relevant numerical optimization based on the Schwarz-Christoffel transformation occurred years ago [21]. Precise inequalities relating radii and various points have also been formulated [22].

The same phrase is used to denote yet another triangle center in [23]. Starting from such a location, a particle undergoing Brownian motion is equally likely to exit through any of the triangle sides. As far as is known, this topic is distinct from our study. Certain integrals and series in [23] deserve greater attention.

## 6. ACKNOWLEDGEMENTS

I am grateful to Thomas Ransford [3, 4] for providing the expression for  $f_w$  involving the Weierstrass sigma function, along with detailed proofs of theorems and answers to several questions.



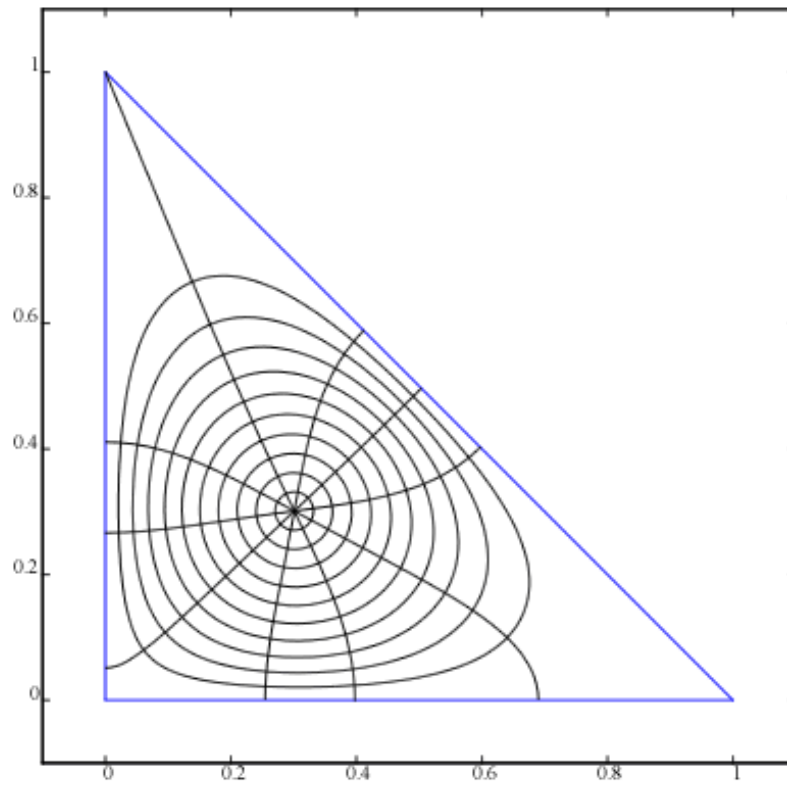


Figure 1: Images of ten concentric circles, center at  $0.301 + (0.301)i$ .

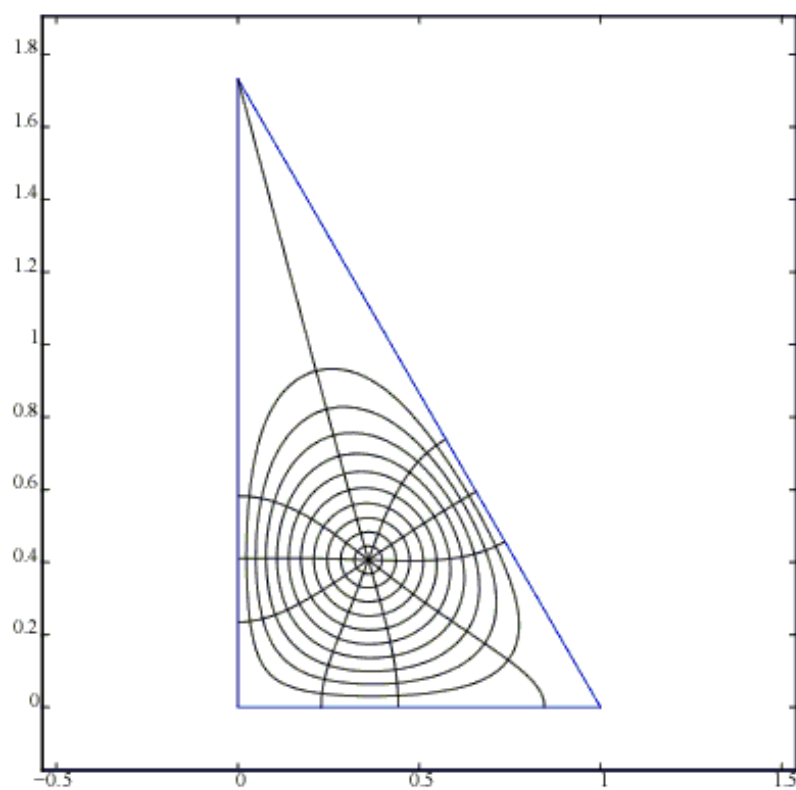


Figure 2: Images of ten concentric circles, center at  $0.359 + (0.406)i$ .

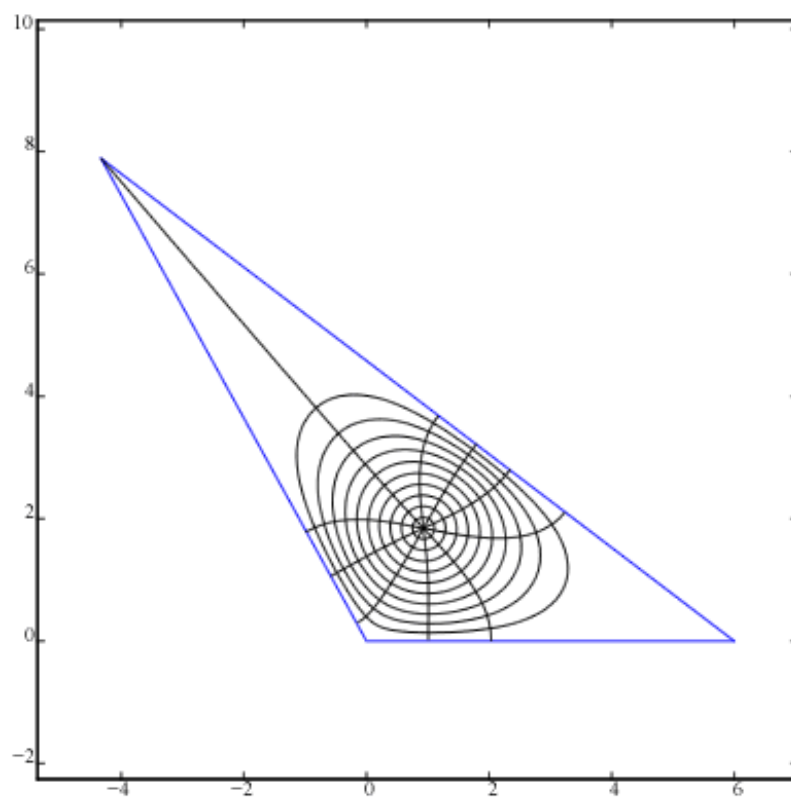


Figure 3: Images of ten concentric circles, center at  $0.929 + (1.842)i$ .

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